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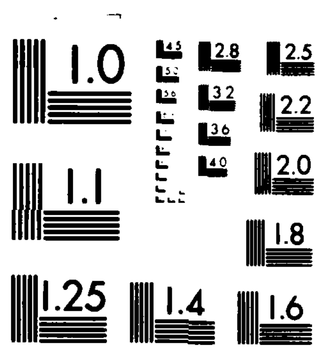
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MEMORANDUM REPORT BRL-MR-3390

NONLINEAR WAVES IN A ROD: RESULTS FOR IMCOMPRESSIBLE ELASTIC MATERIALS

Thomas W. Wright

October 1984

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A nonlinear intrinsic theory is used to describe the motions of a straight round elastic rod including the influence of radial shear and inertia. Consideration of steady wave motions reduces the two coupled partial differential equations to ordinary differential equations for which two integrals of the motion may be found. For incompressible elastic materials with the restriction of small strain gradients, but arbitrary finite strains, a large variety of exact solutions may be found by quadrature. These include large amplitude

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) periodic waves (which may contain shocks), solitary waves, and in some cases waves that are transitional from one stress level to another. Such solutions may be found for uniform stress/strain curves that are concave up or down, that contain inflections, and even curves that represent phase transitions.

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I. INTRODUCTION

A rod may be represented intrinsically as a stretching line endowed with structure.[1] In the most elementary version of this idea one scalar internal variable can be used to simulate the effects of finite transverse dimensions in a straight cylindrical rod that undergoes only axisymmetric motions. The rod then has two kinematically independent degrees of freedom. In cylindrical coordinates R,Θ,Z the motion of the rod is assumed to be

$$z = Z + w(Z,t), \quad r = R[1 + u(Z,t)], \quad \theta = \Theta , \tag{1}$$

where the initial radius of the rod is a, and t is time. All field quantities depend only on the axial coordinate Z and time t.

In a previous paper[2] some of the consequences of the theory were worked out for an elastic material with a strain energy $W(\varepsilon,u,q)$ that depends on axial strain $\varepsilon = w_Z$, radial strain u, and gradient of radial strain $q = u_Z$, where the subscripts denote partial differentiation. In the terms of general continuum mechanics W is the energy density of a one-dimensional elastic continuum with one scalar internal variable, and because of invariance considerations, it is required to be an even function in q.[2] With a kinetic energy density $K = \frac{1}{2}\rho_1 w_t^2 + \frac{1}{2}\rho_2 u_t^2$ Hamilton's principle leads to the following Euler-Lagrange equations,

$$S_Z = \rho_1 w_{tt} , \quad Q_Z - P = \rho_2 u_{tt} \tag{2}$$

where $S = W_\varepsilon$, $P = W_u$, $Q = W_q$. In Reference 2 it was shown that the forces S, P, Q may be interpreted as averages of Piola-Kirchhoff stresses, taken over a cross-section of the rod,

$$S \leftrightarrow \frac{1}{A}\int T^{zZ} dA, \quad P \leftrightarrow \frac{1}{A}\int (T^{rR} + T^{\theta}{}_{\Theta}) dA , \tag{3}$$

$$Q \leftrightarrow \frac{1}{A}\int R T^{rZ} dA ,$$

and the appropriate kinetic energy densities are $\rho_1 = \rho$ where ρ is the density of the rod material and $\rho_2 = \frac{1}{2}\rho a^2$. Equation $(2)_1$ describes forces

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[1] *Antman, S. S., "The Theory of Rods,"* <u>Handbuch der Physik</u>, *Vol. VIa/2, Springer-Verlag, New York 1972.*

[2] *Wright, T. W., "Nonlinear Waves in Rods," in* <u>Proc. IUTAM Symp. on Finite Elasticity</u>, *D. E. Carlson and R. T. Shield, eds., Martinus-Nijhoff Publ., The Hague, 1981.*

and motions in the axial direction, and equation $(2)_2$ describes forces and motions in the radial direction.

II. INCOMPRESSIBLE MATERIAL

If the material is incompressible, then ε and u are no longer kinematically independent and therefore cannot be varied independently in Hamilton's principle. The incompressibility relationship between ε and u, shown in equation (4),

$$(1 + \varepsilon)(1 + u)^2 = 1 \ , \tag{4}$$

must be used as a side condition in the variational principle, and the explicit dependence of the strain energy on radial strain must be suppressed,

$$W(\varepsilon,u,q) = \hat{W}(\varepsilon,q) \ . \tag{5}$$

Under these conditions the Euler-Lagrange equations become

$$S_Z - [\lambda(1 + u)^2]_Z = \rho_1 w_{tt} \ , \ Q_Z + 2\lambda(1 + u)(1 + \varepsilon) = \rho_2 u_{tt}, \tag{6}$$

where λ is a Lagrange multiplier, and now $S = \hat{W}_\varepsilon$ and $Q = \hat{W}_q$. Clearly λ may be interpreted as a superimposed hydrostatic (Cauchy) pressure. In the formulation leading to either (2) or (6) more or less equal weight is given to both radial shear, which enters through the dependence of W on q, and to radial inertia, which enters from the finite value for ρ_2.

III. STEADY WAVES

A steady wave is a disturbance that travels down the rod at a constant speed without distortion. To examine equations (6) for the possibility of steady waves it is supposed that all field variables depend only on the combination $\xi = Z - ct$, where c is an arbitrary constant. The partial differential equations then become ordinary differential equations.

$$S' - [\lambda(1 + u)^2]' = \rho c^2 \varepsilon' \tag{7}$$

$$Q' + 2\lambda(1 + u)(1 + \varepsilon) = \frac{1}{2} \rho a^2 c^2 u'' \tag{8}$$

where the dash signifies differentiation with respect to ξ, and ρ_1 and ρ_2 have been written in terms of ρ. As in Reference 2 there are two integrals of the motion,

$$S - \lambda(1 + \varepsilon)^{-1} = \rho c^2(1 + \varepsilon) + A\ , \tag{9}$$

$$Qu' + S(1 + \varepsilon) - \lambda - \hat{W} = \tag{10}$$

$$\frac{1}{2}\rho c^2[(1 + \varepsilon)^2 + \frac{1}{2}a^2 u'^2] + B\ ,$$

where A and B are constants of integration. The first integral is straight forward. The second integral is obtained by multiplying (7) by $(1 + \varepsilon)$, (8) by u', adding the two resulting equations, and noting that (10) is the integral of the summed expression. Equation (9) may be used to eliminate λ from (10), arriving finally at equation (11),

$$Qu' + A(1 + \varepsilon) - \hat{W} = \frac{1}{2}\rho c^2[\frac{1}{2}a^2u'^2 - (1 + \varepsilon)^2] + B\ . \tag{11}$$

So far the results are exact given the initial premises concerning the strain energy, the kinetic energy, and the condition of incompressibility.

IV. SMALL SURFACE ANGLE

Now, suppose that the surface angle is small in the sense that $au' \ll 1$, and express the strain energy as the first two terms of a power series in u' with coefficients that depend on ε in an arbitrary way,

$$\hat{W}(\varepsilon,q) = \hat{W}(\varepsilon,0) + \frac{1}{2}q^2\ \hat{W}_{qq}(\varepsilon,0) + \ldots\ . \tag{12}$$

Alternatively, consider the special case when $\hat{W}$ can be expressed exactly as

$$\hat{W}(\varepsilon,q) = \int_o^\varepsilon T(\varepsilon)d\varepsilon + \frac{1}{4}a^2\mu(\varepsilon)q^2\ . \tag{13}$$

$T(\varepsilon)$ is engineering stress (force per unit original area) for uniform extension, and $\mu(\varepsilon)$ is interpreted as the shear modulus. (In the linear version of the theory μ would be the bulk shear modulus as in Reference 2, so it seems fair to give it that name for finite extension as well.) From the assumed form of the energy (13) with λ eliminated by use of (8) and u eliminated by use of the incompressibility condition (4), the integral (9) may be written

$$T_o = T(\varepsilon) - \rho c^2(\varepsilon - \varepsilon_o) - \frac{1}{2}\gamma_\varepsilon \varepsilon'^2 - \gamma(\varepsilon)\varepsilon''\ , \tag{14}$$

where $\gamma = \frac{1}{8}a^2\ (1 + \varepsilon)^{-3}[\mu(\varepsilon) - \rho c^2]$, and $T_o = T(\varepsilon_o)$. For convenience A has been replaced by $T_o - \rho c^2(1 + \varepsilon_o)$. Similarly the integral (11) may now be written

$$\frac{1}{4}a^2(\mu - \rho c^2)u'^2 =$$

$$\int_o^\varepsilon T d\varepsilon - [T_o - \rho c^2(1 + \varepsilon_o)](1 + \varepsilon) - \frac{1}{2}\rho c^2(1 + \varepsilon)^2 + B. \tag{15}$$

By rearranging terms and using the incompressibility condition (4), the integral may be rewritten as

$$\frac{1}{16} a^2 \frac{(\mu - \rho c^2)}{(1 + \epsilon)^3} \epsilon'^2 = \int_{\epsilon o}^{\epsilon} [T(\epsilon) - T_o - \rho c^2(\epsilon - \epsilon_o)] d\epsilon + C , \tag{16}$$

where C is an arbitrary constant. Equation (16) also follows from (14) directly upon noting that multiplication by ϵ' makes the terms $\frac{1}{2}\gamma_\epsilon \epsilon'^3 + \gamma\epsilon''\epsilon'$ the exact derivative of $\frac{1}{2}\gamma(\epsilon)\epsilon'^2$. Solution of (16) then follows simply by quadrature.† The integral on the right hand side of (16) is the area between the homogeneous extension curve and a straight line through the point (T_o, ϵ_o) with slope ρc^2. As shown in Figure 1, the cross-hatched areas both to the right and to the left of (T_o, ϵ_o) are positive. The Lagrange multiplier is given by the following expression.

$$\lambda = \frac{1}{8}\left\{\left[\frac{\mu_\epsilon}{(1 + \epsilon)^2} - \frac{3}{2}\frac{\mu - \rho c^2}{(1 + \epsilon)^3}\right]\epsilon'^2 + \frac{\mu \quad \rho c^2}{(1 + \epsilon)^2}\epsilon''\right\} . \tag{17}$$

V. SHOCK WAVES

Although integration of (16) is straightforward, the characteristics of a particular solution depend on the parameters in the problem and the nature of the functions $T(\epsilon)$ and $\mu(\epsilon)$. In addition, it is also possible for discontinuities to occur in certain limiting cases. Therefore, before sketching out the rich variety of solutions available, it is necessary to consider the propagation of shock waves.

Since the cross sectional area of the rod cannot change discontinuously, the radial strain must be continuous according to (1), and since the material is incompressible, the axial strain must be continuous as well.

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†*Aifantis and Serrin*[5] *in discussing phase transitions in the presence of surface tension and Coleman*[6] *in developing a static theory of necking and drawing of fibers have encountered equations of the form*

$T_o = T(\epsilon) + \beta(\epsilon)\,'^2 + \gamma(\epsilon)\epsilon''$. *In fact, for static cases when* $c = 0$, *equation (14) coincides exactly with Coleman's result if his equilibrium equation is obtained from a variational principle (see equation (3.20) and the following discussion in Reference 6). The more general equation was solved by finding an integrating factor (e.g., see Reference 6, equation (3.3)) to obtain a first integral, followed by a quadrature to obtain a second integral.*

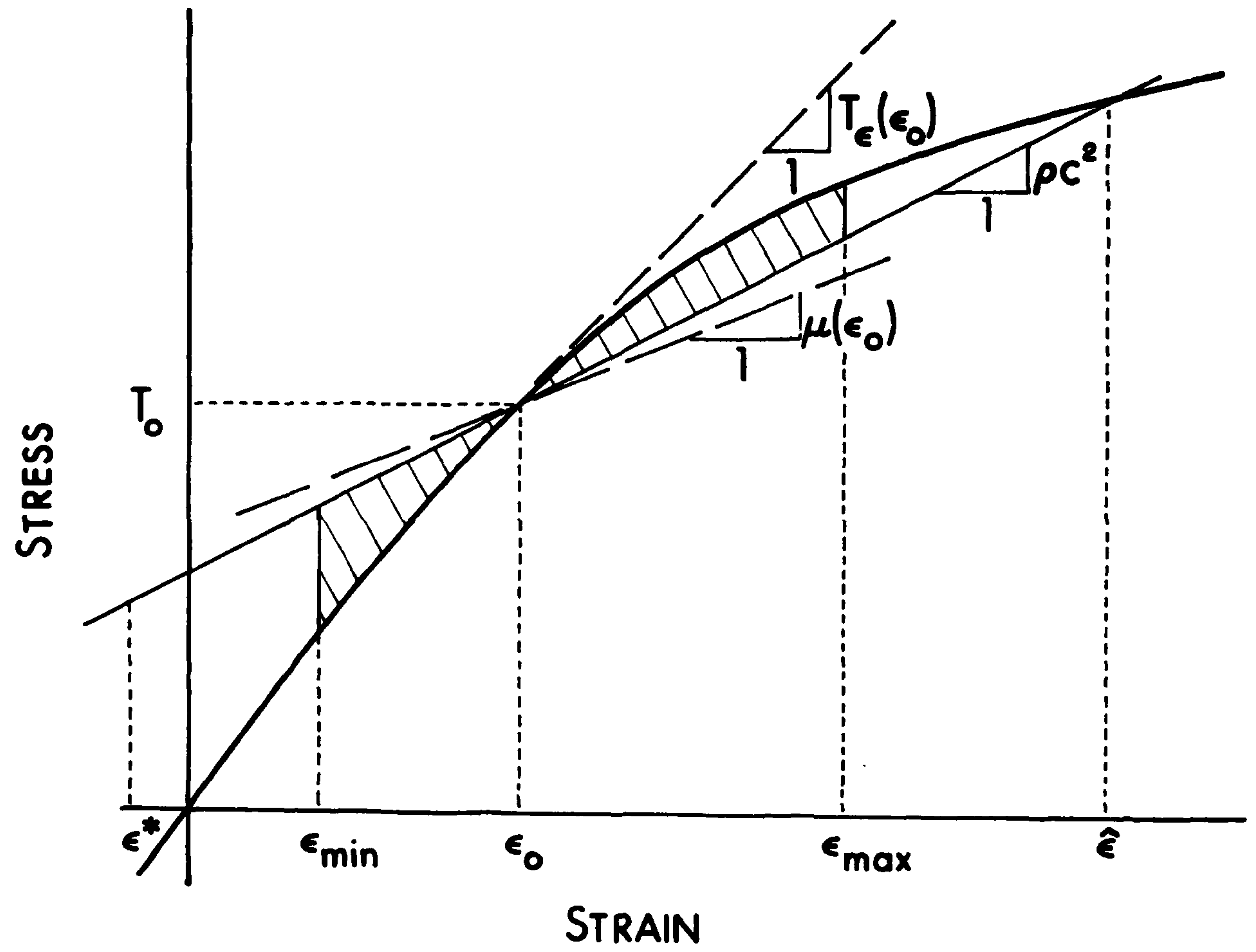

Figure 1. Stress-strain curve and the equal area construction for bounded smooth waves where $\mu(\varepsilon_o)$ [illegible] $^2 < T_\varepsilon(\varepsilon_o)$.

However, both u' and ε' may have discontinuities, and therefore, by (17) so can λ. The appropriate jump condition comes from the integral form of (2) or (6),[3,4]

$$\frac{1}{2}\rho a^2 V^2 = \frac{[\partial \hat{W}/\partial q]}{[q]} \quad , \tag{18}$$

where $[\cdot]$ signifies the jump in a quantity across a shock wave.‡ Substitution for W from (13) gives

$$\rho V^2 = \mu(\varepsilon) \ , \quad [q] \text{ is arbitrary,} \tag{19}$$

so that the speed V of a shock wave of any amplitude is determined solely by the instantaneous shear modulus.

If the shock is incorporated into a steady wave, then $V = c$, and if $\mu(\varepsilon)$ is a monotonic function either increasing or decreasing (the only cases considered in this paper), then for a given steady wave speed c, (19) can be satisfied only at isolated values of ε, say $\varepsilon = \varepsilon^*$, unless μ is truly constant. The situation is shown in Figure 2. Since the left hand side of (16) is zero at a shock wave, no matter what the value of ε'^2, two steady waves with the same speed may be joined together at the shock, but of course each wave has distinct values for T_o, ε_o, and the constant C. These values are not independent in this case, and in fact if one is known for a particular steady wave, the others may be easily computed. This is best seen by referring to Figure 3. For the case shown $\varepsilon > \varepsilon^*$, the point R has coordinates (T_o, ε_o), and $C = -$ Area (PQR). If a second steady wave has the same speed so that Q'R' is parallel to QR, then R' has coordinates (T_o', ε_o'), and $C' = -$Area (PQ'R'). The quantity $[\varepsilon'^2]$ may be determined from (14), noting that $\gamma = 0$ for a steady shock wave.

$$[T_o + \rho c^2(\varepsilon^* - \varepsilon_o)] = -\frac{1}{16}\, a^2 \, \frac{\mu_\varepsilon(\varepsilon^*)}{(1 + \varepsilon^*)^3}\, [\varepsilon'^2] \ . \tag{20}$$

The magnitude of the left hand side of (20) is just the vertical distance QQ' in Figure 3.

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[3] *Nunziato, J. W., and Walsh, E. K., "One-dimensional Shock Waves in Uniformly Distributed Granular Materials," Int. J. Solids and Structures 14 (1978) 681-689.*

[4] *Wright, T. W., "Weak Shocks and Steady Waves in a Nonlinear Rod or Granular Material," (to appear, Int. J. Solids and Structures).*

‡ *In a three-dimensional context, a discontinuity in the strain gradient would be classified as an acceleration wave, but in the present context, where the discontinuity occurs in derivatives of lower order than the highest that appear in the governing differential equation (2) and (8), it is legitimate to use the word shock.*

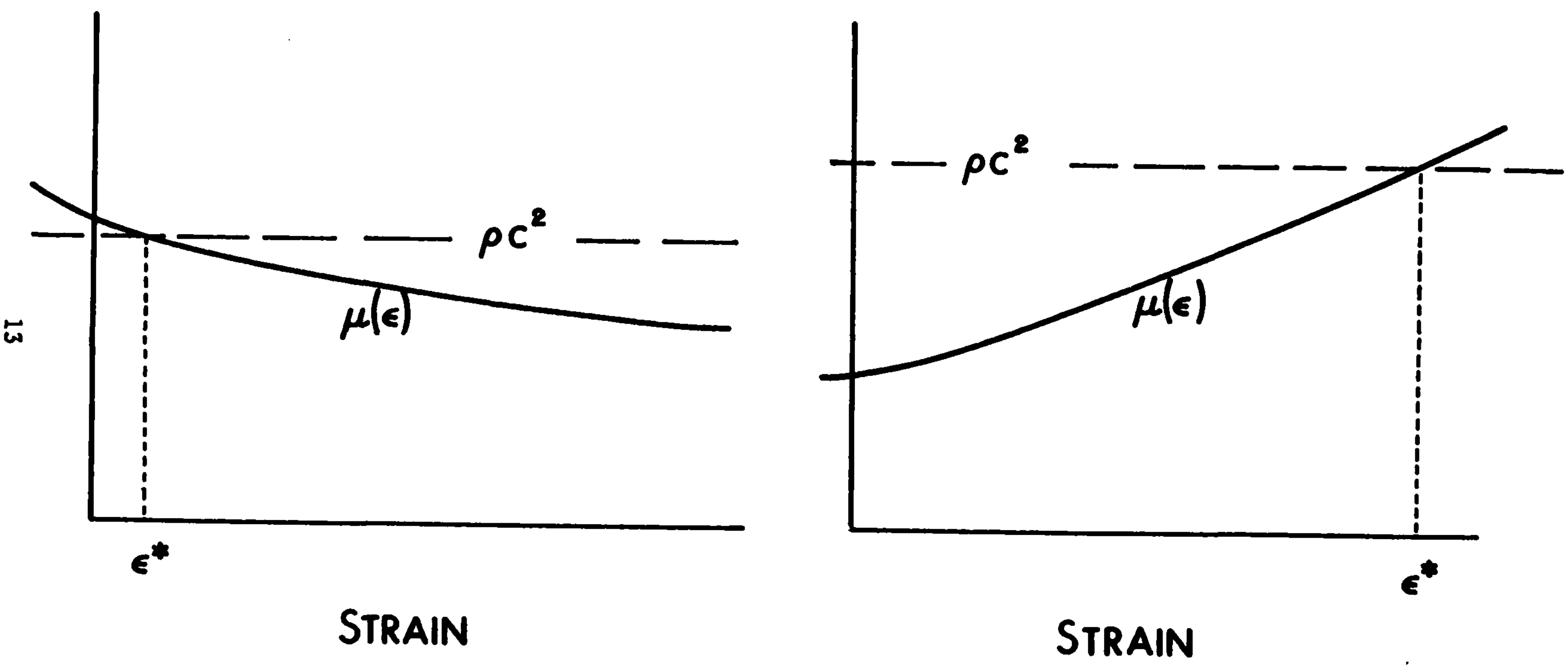

Figure 2. The shear modulus $\mu(\epsilon)$ is assumed to be either monotonically decreasing or monotonically increasing.

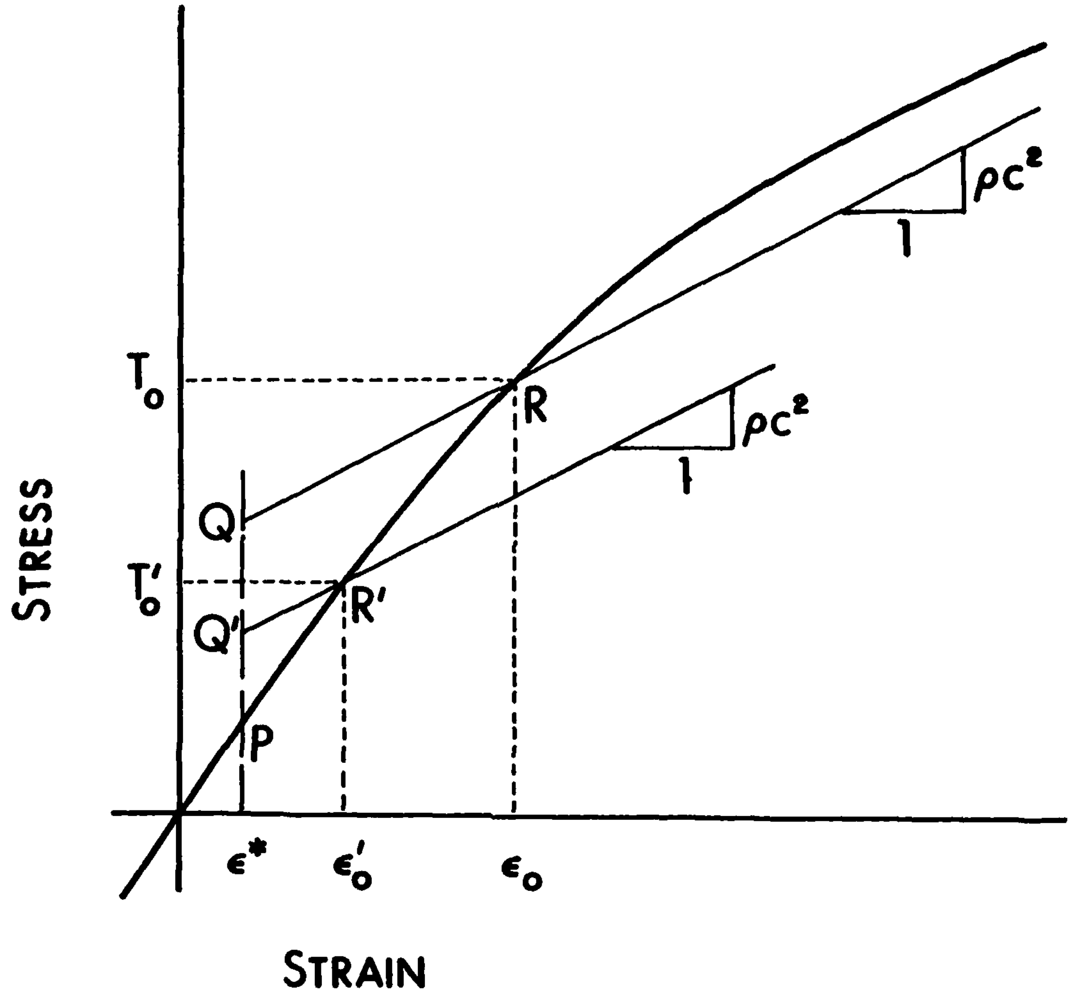

Figure 3. Stress-strain curve and the geometry of shock waves. See text, Section 5.

VI. STEADY WAVE SOLUTIONS

By a process similar to that used in References 5 and 6 it is possible to show that bounded smooth solutions in an infinite rod must be one of three kinds: periodic waves, solitary waves with either a single propagating bulge or neck, or a propagating smooth transition from one state of uniform strain to another. But in addition, solutions with discontinuities, as described in the previous section, may occur. Most of these will appear as limiting cases of the smooth solutions. At this point it is easiest to proceed by considering examples and special cases. Figure 4 shows a sketch of the rod configuration for each of the cases considered.

A. $T(\varepsilon)$ is concave down and at ε_o, $\mu(\varepsilon_o) < T_\varepsilon(\varepsilon_o)$.

Refer to Figure 1 and consider the following construction. Through the point T_o, ε_o draw a straight line with positive slope ρc^2, chosen such that $\mu(\varepsilon_o) < \rho c^2 < T_\varepsilon(\varepsilon_o)$. From Figure 2 find ε^*, which will lie to the left of ε_o if $\mu(\varepsilon)$ is a decreasing function or to the right of ε_o if $\mu(\varepsilon)$ is an increasing function. Finally mark off equal cross hatched areas to the right and left of ε_o with ε^* lying outside the cross hatched interval, and set the constant C equal to minus one half the total cross hatched area. From the construction, it is clear that $\varepsilon'^2 \geq 0$ over the whole interval from ε_{min} to ε_{max} , and $\varepsilon' = 0$ only at the end points. Furthermore, the construction is such that near either end point ε' varies as the square root of the distance from the end point. Every construction of this kind corresponds to a periodic steady wave. As the magnitude of C increases, the cross hatched areas extend farther from ε_o, corresponding to a larger amplitude of the wave, until one of the following finally occurs: either i) $\varepsilon_{max} = \hat{\varepsilon}$, or ii) $\varepsilon_{min} = \varepsilon^*$ for the case of decreasing μ, or iii) $\varepsilon_{max} = \varepsilon^*$ for the case of increasing μ. The preceding discussion shows that periodic waves form a family with three parameters, say T_o, c, and ε_{max} . T_o may be chosen arbitrarily, but c and ε_{max} can only be chosen within limits.

The limiting cases require special treatment.

i) Solitary Waves, $\varepsilon_{max} = \hat{\varepsilon}$. As ε_{max} increases towards $\hat{\varepsilon}$, the period of the wave also increases and becomes infinite when $\varepsilon_{max} = \hat{\varepsilon}$. The wave in

[5] *Aifantis, E. C., and Serrin, J. B., "Towards a Mechanical Theory of Phase Transformations," Tech. Rpt., Corrosion Research Center, University of Minnesota, Minneapolis, 1982.*

[6] *Coleman, B. D., "Necking and Drawing in Polymeric Fibers Under Tension," Arch. Rat. Mech. Anal. 83 (1983) 115-137.*

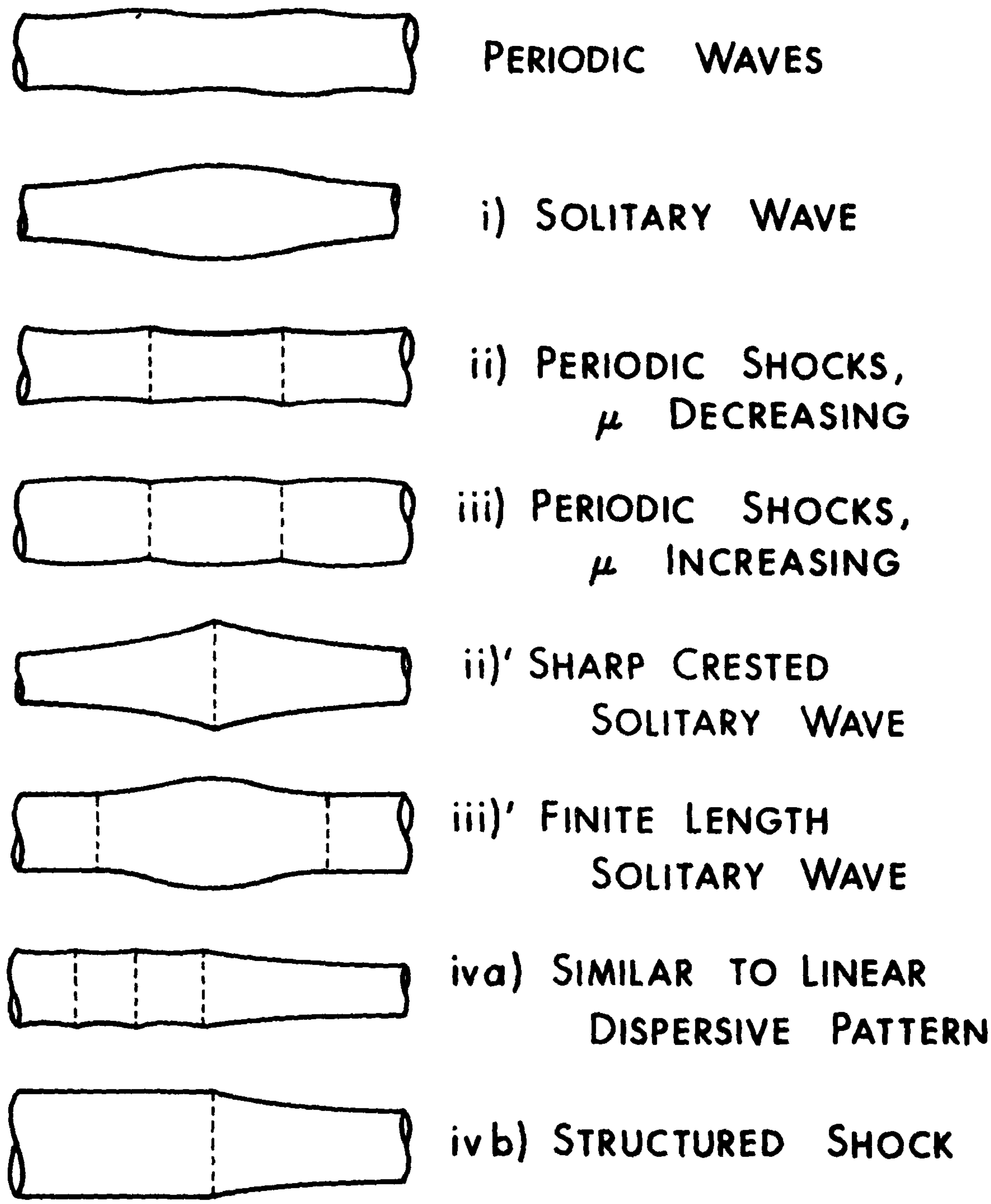

Figure 4. Typical mode shapes of steady waves. Dashed lines indicate shock waves. Roman numerals refer to cases discussed in Section 6.

this case consists of a single crest, a propagating bulge in the radius of the rod for the stress/strain curve shown in Figure 1. At infinity behind and ahead of the wave the strain is $\hat{\varepsilon}$, decreasing to ε_{min}, as determined by the equal area rule, at the crest. Since the maximum strain is determined by the choice of T_o and c, solitary waves form a two parameter family.

ii) Shock Waves, $\varepsilon_{min} = \varepsilon^*$. As ε_{min} decreases towards ε^*, the crests of the wave become less and less rounded until finally when $\varepsilon_{min} = \varepsilon^*$, the crests become sharp with ε' changing discontinuously. Since C in (16) has been chosen such that both sides vanish when $\varepsilon = \varepsilon^*$,

$$\lim_{\varepsilon \to \varepsilon^*} \varepsilon'^2 = \frac{16(1 + \varepsilon^*)^3}{a^2} \cdot \frac{T(\varepsilon^*) - T_o - \rho c^2(\varepsilon^* - \varepsilon_o)}{\mu_\varepsilon(\varepsilon^*)} , \tag{21}$$

and ε' jumps from one of the values that satisfies (21) to the other. The numerator in the second factor of (21) is just the negative of the vertical distance PQ in Figure 3. Choice of c determines ε^*, and choice of T_o determines ε_{max} from the equal area rule, so waves of this kind form a two parameter family.

iii) Shock Waves, $\varepsilon_{max} = \varepsilon^*$. This case is the same as ii) except that *shock waves* occur at ε_{max} rather than ε_{min}.

It may happen that two of the limiting cases occur simultaneously.

ii)' Shock Wave, $\varepsilon_{max} = \hat{\varepsilon}$ and $\varepsilon_{min} = \varepsilon^*$. In this case the solitary wave has a sharp crest on it. The two cross hatched areas in Figure 1 extend all the way from ε_o to $\hat{\varepsilon}$ and from ε_o to ε^*. Since only one choice of T_o will make the areas equal for a given c, these waves form a one parameter family.

iii)' $\varepsilon_{max} = \hat{\varepsilon} = \varepsilon^*$. By analogy with the previous case it might seem that this case would correspond to a single shock wave occurring at the maximum strain, but in fact ε' is zero at ε_{max}. This occurs because the right side of (16) now has a double zero at ε_{max}, but the coefficient of ε'^2 on the left side has only a simple zero. Thus, these waves are simply periodic waves of maximum wave length. However, since discontinuities can occur at ε^*, one cycle of a wave could be joined to a uniform state both behind and ahead of the wave thus creating a solitary wave of finite wavelength. Since the choice of c determines ε^* (and hence ε_{max}) and the equal area construction will determine all other parameters, these waves are a one parameter family.

iv) As stated in Section 5, a shock wave may join two steady waves together if they have the same speed. Cases ii), ii)', and iii) suggest many ways in which this could be done. Two sharp crested periodic waves with unequal wave lengths could be joined, for example, but two more interesting cases are a) a sharp crested periodic wave jointed to a sharp crested solitary wave, which resembles the familiar dispersive pattern in linear elasticity for a step load on the end of a right circular cylinder, and b) a uniform state joined to a sharp crested solitary wave, which is really just a limiting form of iva) and might be termed a structured shock wave. With these constructions it is possible to join two arbitrary (within limits) stress states with a steady wave.

B. $T(\varepsilon)$ is concave down and at ε_o, $T_\varepsilon(\varepsilon_o) < \mu(\varepsilon_o)$.

If the stress/strain curve for uniform extension turns over so far that $T_\varepsilon(\varepsilon_o) < \mu(\varepsilon_o)$, then the previous construction for bounded solutions must be modified by choosing c such that $T_\varepsilon(\varepsilon_o) < \rho c^2 < \mu(\varepsilon_o)$. In all other respects, however, the construction is the same. Equal areas to the right and left of ε_o extend to ε_{max} and ε_{min}. Limiting cases occur when ε_{min} coincides with either $\hat{\varepsilon}$ or ε^* (for the case of increasing μ) or when ε_{max} coincides with ε^* (for the case of decreasing μ). Periodic waves, including periodic shock waves, look much the same as before, but the wave of infinite period (i.e., the solitary wave) is now a propagating neck rather than a bulge as previously and the limiting case is a sharp notch rather than a sharp crest. Since two different steady waves can be joined through a shock wave, many combinations are possible, but perhaps the most interesting case is the structured shock wave which joins a uniform state to half of a sharp notch. The sketches in Figure 5 should be compared with those in Figure 4.

C. $T(\varepsilon)$ has an inflection point and $T_\varepsilon(\varepsilon_o) < \mu(\varepsilon_o)$.

In this case the equal area construction can lead to a smooth transition from one uniform state to another provided that $T_\varepsilon(\varepsilon_o) < \rho c^2 < \mu(\varepsilon_o)$ and ε^* lies outside the range of strains in the transition. The constant C is equal to half the doubly cross hatched area shown in Figure 6a. The construction is valid even if the stress/strain curve for uniform extension has a maximum and a minimum as shown in Figure 6b. Such a curve has been suggested by Ericksen as a model for phase change[7] and was used by Coleman to describe necking and drawing of polymers.[6] In the latter context the solution represented by Figure 6b shows the influence of radial inertia and radial shear on the drawing process at high speed. Note also the implication that the drawing speed is limited by the shear speed of the undrawn (drawn) material for $\mu(\varepsilon)$ increasing (decreasing).

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[7] *Ericksen, J. L., "Equilibrium of Bars,"* *J. Elasticity* *5 (1975) 191-201.*

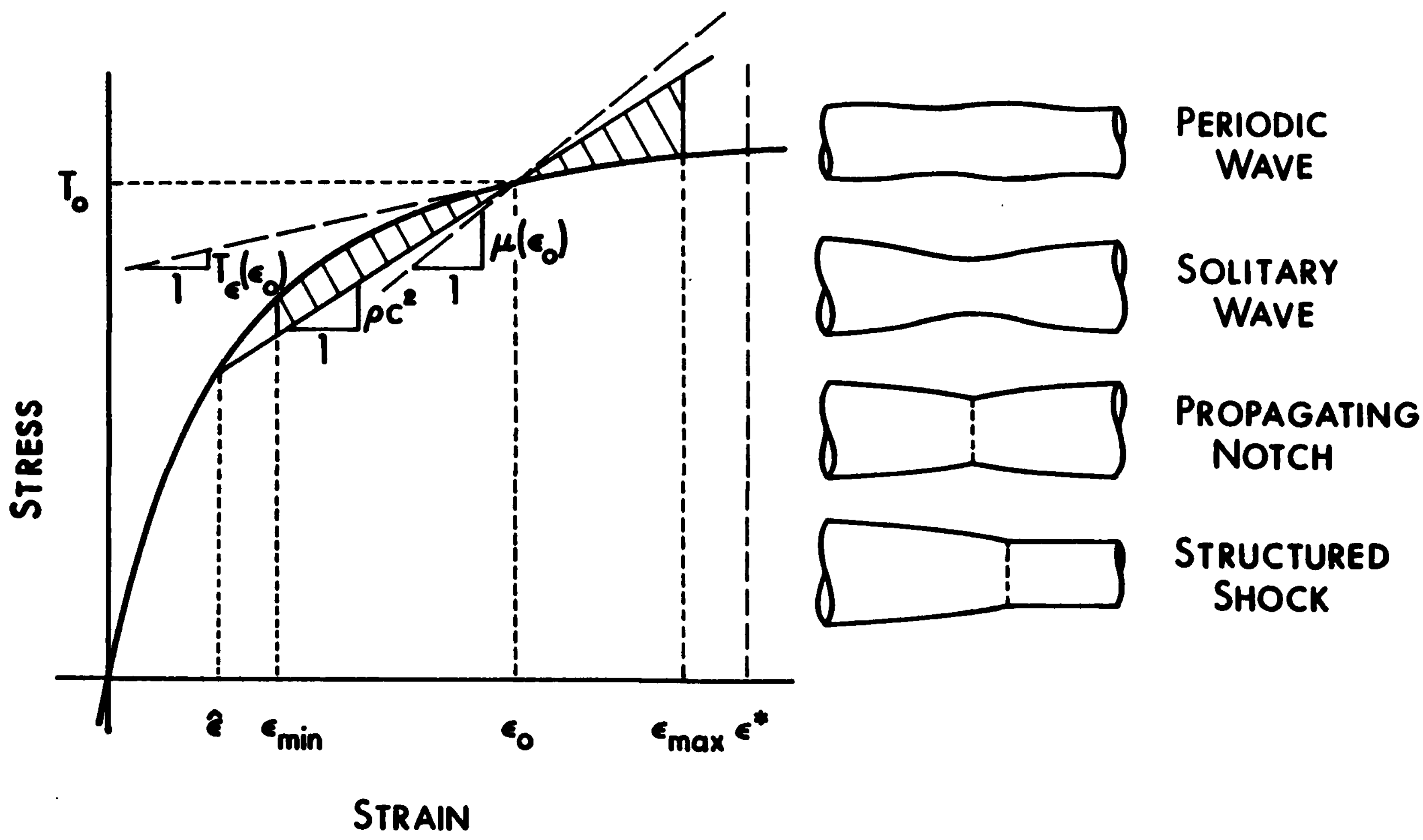

Figure 5. Stress strain curve and the equal area construction for the case $T_\varepsilon(\varepsilon_o) < \rho c^2 < \mu(\varepsilon_o)$. Typical mode shapes are also shown. Dashed lines indicate shock waves.

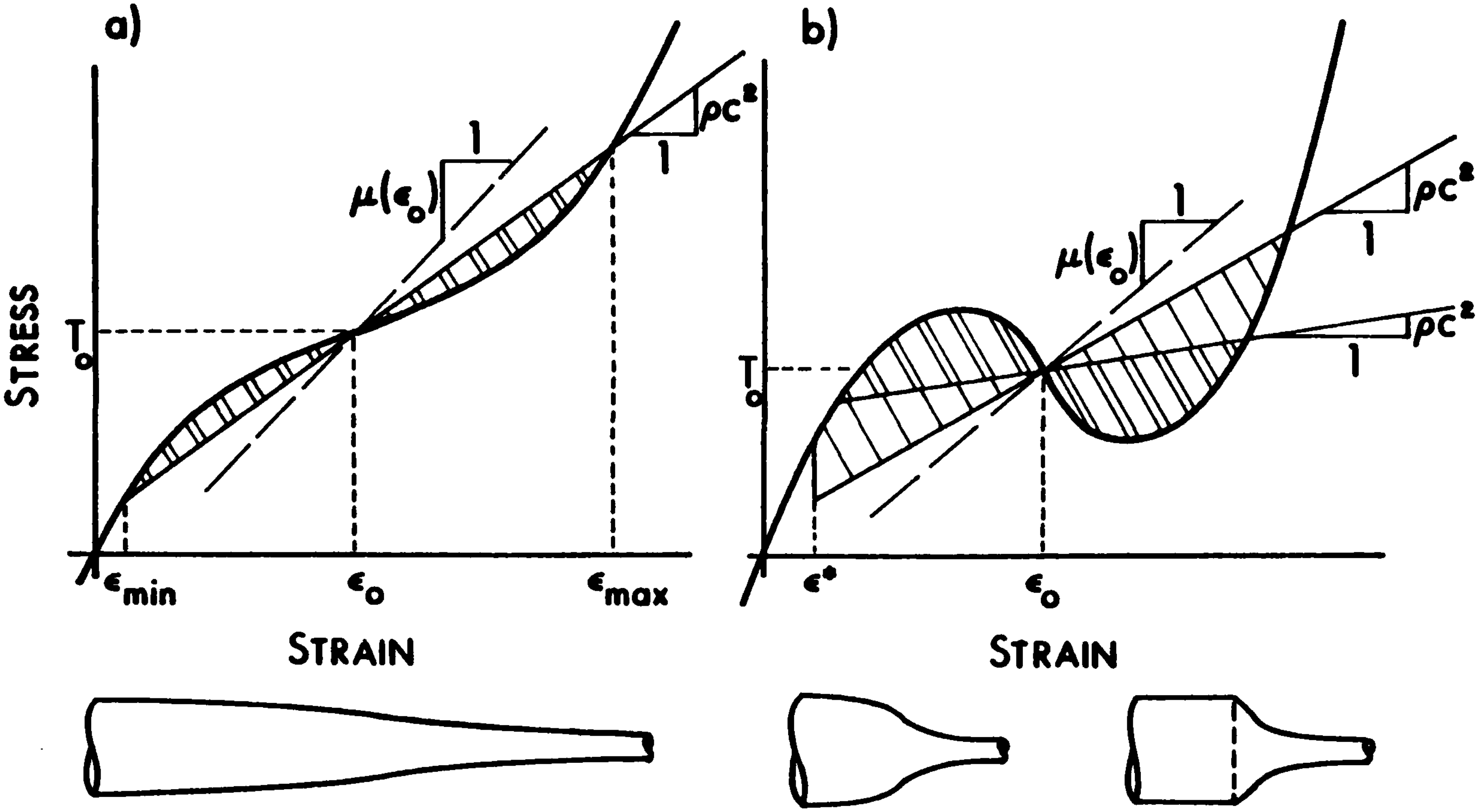

Figure 6. Stress strain curves with inflections and the equal area construction. Case a) shows a simple inflection; Case b) shows a model for a phase change. Typical mode shapes are also shown.

It is possible for ε^* to lie at either the minimum or the maximum strain of the transition. In that case there will be a shock discontinuity on one side or the other of the transition, but as long as the equal area construction can be maintained, a transition solution still exists. In this case the constant C is equal to half the total cross hatched area in Figure 6b.

VII. UNBOUNDED SOLUTIONS

So far only solutions that are bounded for $-\infty < Z - ct < \infty$ have been described, but there are unbounded steady wave solutions as well. To find these, it is easiest to adopt a slightly different point of view based on the phase plane.

As an example refer to Figure 1, consider the case shown there for $\mu(\varepsilon)$ decreasing and $\varepsilon^* < \varepsilon_o$, and suppose that ε_o and c are both fixed. The construction is such that the right hand side of equation (16) has a minimum at $\varepsilon = \varepsilon_o$ and a maximum at $\varepsilon = \hat{\varepsilon}$. The constant C can be either positive or negative so the right hand side of (16) has either one or three real zeros including a possible double zero at the maximum or minimum. The coefficient of ε'^2 on the left hand side of (16) has only a simple zero at $\varepsilon = \varepsilon^*$. Bearing these facts in mind and taking account of the relative positions of the zeros on both sides, one may solve equation (16) for ε' as a function of ε and sketch a family of curves, as in Figure 7, each member of which corresponds to a different value of C. Because of the zero at ε^*, no curve can cross the vertical line $\varepsilon = \varepsilon^*$ except the case where the right hand side of (16) has a zero at ε^* as well, and because of incompressibility, no curve can cross the line $\varepsilon = -1$. For other values of T_o, ε_o, and c the phase plane diagram can be substantially modified. For example, the location of ε^* obviously will have a pronounced effect, and if the straight line through T_o, ε_o has no other intersection with $T(\varepsilon)$, there can be no orbits that correspond to smooth solitary waves.

The closed trajectories around ε_o represent periodic waves with the curve labeled L being the limiting case of sharp crested waves. Examples of unbounded trajectories are labeled A through G. Even though these trajectories are unbounded in the phase plane, the corresponding deformation patterns are readily interpretable, and are also sketched in Figure 7, showing the rod radius as a function of ξ. Although no solution to an initial/boundary value problem could include any point with $\varepsilon = -1$ or $\varepsilon = \infty$ or $\varepsilon' = \pm\infty$, a segment of an unbounded trajectory that does not include such singularities would be perfectly acceptable. Problems in which the material moves at constant speed with respect to fixed boundaries, where fixed boundary conditions are specified, would fall in this category.

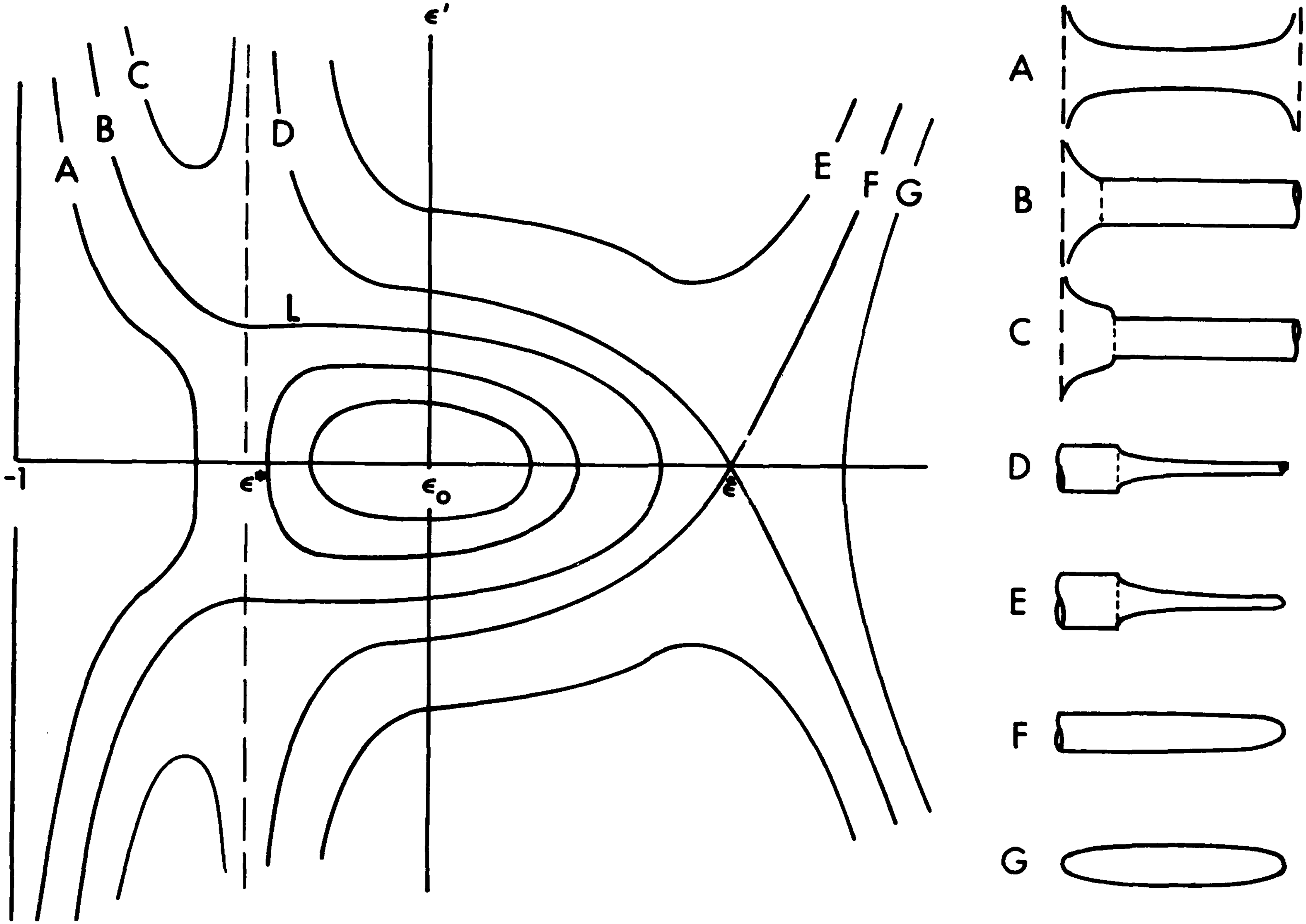

Figure 7. Phase plane showing both bounded and unbounded trajectories. Typical mode shapes for some of the unbounded trajectories are also shown.

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